Lecture 3. ***Concept of observability. Criteria of observability R. Kallman and E. Gilbert***

Methods of synthesis of different control laws generally require complete state vector for their implementation. If some components of state vector are unavailable for measuring, then direct implementation of control laws is impossible. In these cases there is a necessity in ways that would let indirectly evaluate inaccessible or completely unavailable components of state vector for direct measuring. In latest Control theory such ways exist. They are based on using equations of the object motions, which are considered to be known. Devices for measuring coordinates of an object, base of which contains the equations of the object itself, are called observing devices (OD).

Suppose that the investigated ACS is described by the system of equations (5.4), (5.5).

In case, when only output vector Y is available for measuring instead of complete state vector X, it is necessary to find a reasonable substitution (evaluation) of the system state vector. To evaluate the state vector it is reasonable to use information on input and output variables of the system and on its structure (dimensions of matrixes *A, B, C*). If the rank of the matrix of observations *C* is equal to order of the system $rank(C) = n$, i.e. $r = n$, then to define a state vector *X* is possible through algebraic operations by using *only instantaneous values of output vector*. In this case

$$\hat{X} = C^{-1} Y ,$$

where $\hat{X}(n \times 1)$ is estimation of state vector of the system.

In case when $r < n$ such method cannot be applied, as the matrix of observations *C* is degenerated $(\det C = 0)$ and there is no any backward matrix for it. At actual conditions, as a rule, it is $r << n$.

3.1. Concept of observability

Let us introduce the concept of observability. We have description DS in a state-space:

$$\begin{cases} \dot{X} = AX + BU & (5.4) \\ Y = C^T X & (5.5) \end{cases} ,$$

Definition. The system (5.4), (5.5) is called *completely observed*, if initial state $x(0)$ may be defined through values of output variables vector $y(t)$, measured at termination interval of time $0 \le t \le T$.

Let us introduce the criterion of observability, which on basis of values of matrixes *A* and *C* will let define observability of the system. For this let us bring the system (5.4), (5.5) to diagonal form through linear transformation.

By means of linear transformation of coordinates $X = V \cdot X^*$, where matrix *V* is a modal matrix for an initial matrix *A*, we will transform equations of state-space of a system and observation (5.4), (5.5) to a diagonal look:

$$X = VX^*$$

$$V\dot{X}^* = AVX^* + BU$$

$$V^{-1}V\dot{X}^* = V^{-1}AVX^* + V^{-1}BU$$

$$\begin{cases} \dot{X}^* = \Lambda X^* + B^*U \\ Y^* = C^*X^* \end{cases}, \qquad \begin{matrix}(5.6)\\(5.7)\end{matrix}$$

where $\Lambda = V^{-1}AV$, $B^* = V^{-1}B$, $C^* = CV$

Applying representation of the system in diagonal form (5.6), (5.7), we may formulate the following criteria of observability.

The formulation Gilbert's criterion of observability:

If matrix doesn't contain zero columns, the system (5.6), (5.7) is completely observed, otherwise it is non-observed.

The criterion implies that vector of output variables of completely observed system must contain *complete state vector*.

Example 5.8. Suppose that ACS mathematical description is specified as a system of equations (5.6), (5.7), where matrixes Λ, B^* and C^* are equal to the following:

$$\Lambda = \begin{vmatrix} -3 & 0 \\ 0 & -4 \end{vmatrix}, \; B^* = \begin{vmatrix} -3 \\ 4 \end{vmatrix}, \; C^* = \begin{vmatrix} 5 & 6 \end{vmatrix}.$$

It is required to characterize observability of investigated system.

The investigated system is completely controllable, as matrix C^* doesn't contain any zero columns.

Let us write the system in a scalar form and see it more visually

$$\begin{cases} \dot{x}_1^* = -3x_1^* + u_1 \\ \dot{x}_2^* = -4x_2^* + 2u_2 \\ y^* = 5x_1^* + 6x_2^* \end{cases}.$$

The investigated system is completely controllable, as the vector of output coordinates contains the complete state vector.

There exists another criterion of observability. It doesn't require transforming the system equations to a diagonal form, which applies original matrixes *A* and *C*.

Formulation Kalman's criterion of observability:

For the system (5.4), (5.5) to be *completely observed* it is necessary and enough for a rank of block matrix of observability

$$K_{ob}^{T}=\left(C^{T},A^{T}C^{T},\left(A^{T}\right)^{2}C^{T},...,\left(A^{T}\right)^{n-1}C^{T}\right)$$

to be equal to the system order *n*, i.e. $rank\,K_{ob}^{T}=n$.

Form of the matrix of observability implies that its size is equal to $(n\times nr)$, i.e. it has *"n"* of rows and *"nr"* of columns. If from the columns of the matrix of observability K_{ob}^{T} may be made at least one matrix of $(n\times n)$ size, the rank of which is equal to system order *"n"*, then the system is to be completely controllable.

Example 5.9. Suppose that ACS mathematical description is specified as a system of equations (5.4), (5.5), where matrixes *A*, *B* and *C* are equal to the following:

$$A=\begin{vmatrix}6 & 1\\ 4 & 6\end{vmatrix},\ B=\begin{vmatrix}-3\\ 4\end{vmatrix},\ C=\begin{vmatrix}-1 & 2\end{vmatrix}.$$

It is required to characterize observability of investigated system.

Algorithm and solution

1. We should make a block matrix of observability and define its rank.

Matrix $A(2*2)$; hence, $n=2$, the matrix of observability for this system is written in the following way:

$$K_{ob}^{T}=\left(C^{T},A^{T}C^{T}\right).$$

2. A rank of the block matrix of observability is defined:

$$A^{T}C^{T}=\begin{vmatrix}6 & 1\\ 4 & 6\end{vmatrix}\begin{vmatrix}-1\\ 2\end{vmatrix}=\begin{vmatrix}-4 & 8\end{vmatrix};\ K_{ob}^{T}=\begin{vmatrix}-1 & 2\\ 4 & 8\end{vmatrix};\ \Delta_1=1\neq 0;\ \Delta_2=0;$$

$$n=2;\ rank\,K_{ob}^{T}=1\neq n.$$

Conclusion: investigated system is non-observability, as the rank of the block matrix of observability is not equal to the system order.

To evaluate the system observability at $r << n$, it is necessary to make an observing device (OD).

Supposed that $\hat{X}$ is an evaluation of state vector X. If some components of vector X are unavailable for measuring, then it is necessary to make such evaluation for state vector $\hat{X}$, in order $\left|\hat{X} - X\right| \to 0$ is at $t \to \infty$.

Let us view some dynamic system, inputs of which are inputs of the original linear system, and outputs are vector components $\hat{X}$ are estimation of state vector. Let us introduce the next definition of OD.

Definition. Dynamic system which forms at output the estimation of state vector $\hat{X}$ by the data on inputs and outputs of original system is called an observing device.

Schematically CO together with an observing device may be presented as:

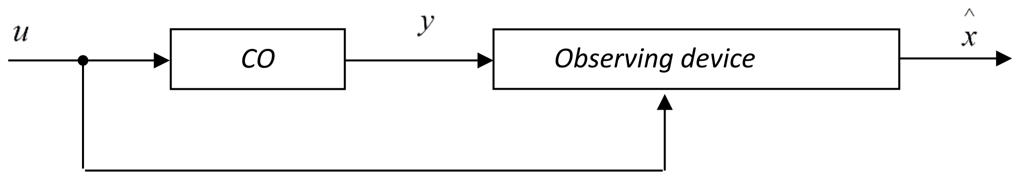

Fig. 5.33. Structural scheme of the system with OD

The simplest way of making OD is direct modeling of dynamics of the controlled object through means of analog and digital computer facilities.